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A GENERAL ALGORITHM FOR COMPRESSIBLE AND INCOMPRESSIBLE FLOW—PART I. THE SPLIT, CHARACTERISTIC-BASED SCHEME

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SUMMARY

The paper outlines the formulation of a novel algorithm which can be used for the solution of both compressible and incompressible Navier–Stokes or Euler equations. Full incompressibility can be dealt with if the algorithm is used in its semi-explicit form and its structure permits arbitrary interpolation functions to be used avoiding the Babuška–Brezzi restriction. In a fully explicit version it introduces a rational form of balancing dissipation avoiding the use of arbitrary parameters and forms for this.

KEY WORDS: compressible flow; CFD (computational fluid dynamics); finite elements in fluids

1 INTRODUCTION

The extensive development of the finite element procedures for the solution of compressible, high speed flow problems occurred only in the last decade. Without doubt this was due to the 'rediscovery' of the Lax–Wendroff methods[1] in the context of finite elements as the Taylor–Galerkin process[2,3] and the introduction of the characteristic Galerkin method.[4–6] However while the former formulation could be used for a general form of conservation equations, typical of Navier–Stokes or Euler problems with multiple variables and characteristic speeds, only the latter, the characteristic Galerkin method, justifies the use of the Galerkin spatial discretization. Both formulations yield an identical approximation only when a single variable and one characteristic speed exists. In the next section of this paper we recall the essentials of the characteristic Galerkin process and its rationale.

The original Taylor–Galerkin stabilization of the finite element discretization has been widely supplemented by the use of empirical artificial diffusion forms, mainly developed in the context of finite difference methods, and the results consequently improved. However, it appeared to the authors that a return to the single characteristic speed, for which the procedures were proved, could be achieved by a suitable operator splitting procedure.

The key to such a split lies in a fractional step method devised originally by Chorin[7,8] and subsequently developed by others[9–19] for incompressible flows. When compressibility exists, the acoustic or compressible wave phenomena, governed now by self-adjoint equations, can be separated from transport and dissipation. The latter processes in turn are governed by systems of equations with the fluid velocity as a single characteristic speed.[20–23]

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The original Chorin method was initially devised for the purpose of implementing a time-stepping process for the momentum and continuity equations in which the essential variables were the flow velocity **u** and pressure p. The process, applicable to incompressible flows and sometimes interpreted as a projection method, starts by obtaining an approximate velocity field using the momentum equation with the pressure gradients omitted. This first step is followed by solving for the unknown pressures on inserting the velocity approximation into the continuity equation. The final stage is the correction of the velocity vector using the computed pressure terms. This led to the process sometimes being known as the velocity correction method.

The method essentially separates the pressure calculation into one involving a Laplacian form which is self-adjoint and only a single characteristic velocity is involved in the first stage, clearly achieving the desired effect. However when the transient form is used for steady-state solution further benefits can arise. One of these, observed by Schneider *et al.*[11] and by Kawahara and Ohmiya,[15] is that the Babŭska Brezzi stability restrictions, well known in the velocity–pressure finite element discretization, no longer apply, as the discrete steady-state equations do not have a zero diagonal term. Now a term proportional to the time increment is there inserted and this allows arbitrary and convenient interpolations to be used for **u** and p. Here for instance any equal interpolation is possible avoiding the difficulties frequently encountered in the use of such interpolation coupled with the previously mentioned Taylor–Galerkin procedures.

The result here is in essence similar to that obtained by Hughes *et al.*[24], Sampaio[25] and the wider interpretation of it described in Reference 26.

These combined merits of the use of the fractional step procedures have been realized by the authors earlier[21] but so far success has been limited to applications in a non-conservative form of equations of Navier–Stokes and Euler[22] or to shallow water equations.[23] In the present paper the approach is considerably modified allowing the full form of conservation equations to be dealt with. Indeed the new approach can be simply extended to deal with the transport of additional variables such as turbulence parameters or chemical reactions.

The essential step of the new procedure is the realization that in each computational step the transport of a single scalar quantity occurs and the treatment of this is described in the next section.

The characteristic Galerkin method, as any other linear scheme, does not preclude the presence of overshoots and undershoots in the vicinity of sharp gradients of the solution. This is true even for the linear convection–diffusion equation. In the numerical simulation of compressible flows, these small oscillations, in principle localized, may deteriorate the global stability of the numerical solution due to the non-linear character of the problem. We describe in Section 5 the shock capturing technique that we use for the compressible Navier–Stokes equations, which is based on previous work for the convection–diffusion equation.[27–28] This technique is similar to that proposed by other authors using different reasoning.[29–31]

2. THE SCALAR CONVECTION–DIFFUSION PROBLEM AND THE CHARACTERISTIC GALERKIN EXPLICIT APPROXIMATION

Before proceeding to the description of the full algorithm we shall recall the application of the characteristic Galerkin method in the explicit form to a typical convection–diffusion process with a scalar-dependent variable ϕ.

The governing equations can here be written always in a conservation form as

$$\frac{\partial \phi}{\partial t} + \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} + Q = 0 \tag{1}$$

where x_i is the ith Cartesian co-ordinate ($i = 1, 2, 3$),

$$F_j = u_j \phi \tag{2a}$$

is the convected flux,

$$G_i = -k \frac{\partial \phi}{\partial x_i} \tag{2b}$$

is the diffusion flux,

$$Q = Q(\mathbf{x}, t) \tag{2c}$$

is the source term, and

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \tag{2d}$$

with components u_i is the velocity field, which is assumed to be known.

The full equation can thus be alternatively written as

$$\frac{\partial \phi}{\partial t} + u_i \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i}\left(k \frac{\partial \phi}{\partial x_i}\right) + a\phi + Q = 0 \tag{3}$$

where $a = \partial u_i / \partial x_i$.

Let us denote by $\tilde{\mathbf{x}}(\mathbf{x}_{\text{ref}}, t_{\text{ref}}; t)$ the trajectory of the particle that at time $t = t_{\text{ref}}$ is located at the spatial point $\mathbf{x}_{\text{ref}}$, so that $\tilde{\mathbf{x}}(\mathbf{x}_{\text{ref}}, t_{\text{ref}}; t_{\text{ref}}) = \mathbf{x}_{\text{ref}}$. This trajectory, or characteristic, will be the solution of the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\mathbf{x}}(t) = \mathbf{u}(\tilde{\mathbf{x}}(t), t) \tag{4a}$$

$$\tilde{\mathbf{x}}(t_{\text{ref}}) = \mathbf{x}_{\text{ref}} \tag{4b}$$

In the short-hand notation $\tilde{\mathbf{x}}(t)$ it is understood that $\tilde{\mathbf{x}}$ depends also on t_{ref} and $\mathbf{x}_{\text{ref}}$ through the boundary condition (4b). We have that

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(\tilde{\mathbf{x}}(t), t)|_{t = t_{\text{ref}}} = \left(\frac{\partial \phi}{\partial t} + u_i \frac{\partial \phi}{\partial x_i}\right)\Bigg|_{\mathbf{x} = \mathbf{x}_{\text{ref}},\, t = t_{\text{ref}}} \tag{5}$$

and therefore equation (3) may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(\tilde{\mathbf{x}}(t), t) - \frac{\partial}{\partial x_i}\left(k \frac{\partial \phi}{\partial x_i}\right) + a\phi + Q = 0 \tag{6}$$

where all the terms are understood to be evaluated at $\mathbf{x} = \tilde{\mathbf{x}}(t)$. The idea now is to discretize the derivative $\mathrm{d}/\mathrm{d}t$ using a finite difference scheme, that is, to discretize the total derivative in equation (3) along the characteristics.

It is observed that equation (6) only involves self-adjoint operators in space. In this case, it is known that the standard Galerkin approximation in space is optimal but the inconvenience of a moving co-ordinate system is introduced. To avoid this difficulty a local approximation can be used.[3-6] The best approximation that we can expect using a single-step scheme is second order. This is why we shall first discretize equation (6) in time up to second order. Once this is done, different schemes can be obtained by approximating the second (temporal) argument of ϕ, yielding discrete schemes with a lower temporal accuracy but potentially second-order accurate in space.

Suppose now that we have the solution at time t_n and we want to compute it at time t_{n+1}. Let t_{ref} be a reference time in $[t_n, t_{n+1}]$. The discretization of equation (6) that we consider is:

$$\begin{aligned}
&\frac{1}{\Delta t}[\phi(\tilde{\mathbf{x}}(t_{n+1}), t_{n+1}) - \phi(\tilde{\mathbf{x}}(t_n), t_n)] \\
&\quad - \theta \frac{\partial}{\partial x_i}\left(k \frac{\partial \phi}{\partial x_i}\right)(\tilde{\mathbf{x}}(t_{n+1}), t_{n+1}) - (1-\theta)\frac{\partial}{\partial x_i}\left(k \frac{\partial \phi}{\partial x_i}\right)(\tilde{\mathbf{x}}(t_n), t_n) \\
&\quad + \theta a\phi(\tilde{\mathbf{x}}(t_{n+1}), t_{n+1}) + (1-\theta)a\phi(\tilde{\mathbf{x}}(t_n), t_n) \\
&\quad + \theta Q(\tilde{\mathbf{x}}(t_{n+1}), t_{n+1}) + (1-\theta)Q(\tilde{\mathbf{x}}(t_n), t_n) = 0
\end{aligned} \tag{7}$$

where $\theta \in [0, 1]$. To obtain a second-order approximation we must take $\theta = 1/2$ (Crank–Nicolson scheme).

We derive now an explicit expression for $\phi(\mathbf{x}(t_{n+1}), t_{n+1})$ and $\phi(\tilde{\mathbf{x}}(t_n), t_n)$. This will allow us to obtain a semi-discrete system of equations where all the terms will be evaluated at the same point of the same spatial domain. This can be done in particular for $t_{\text{ref}} = t_n + \Delta t/2 = t_{n+1/2}$ and $t_{\text{ref}} = t_n + \Delta t = t_{n+1}$. The first option yields the classical Crank–Nicolson discretization of equation (3), whereas the second introduces some additional terms that enhance the stability of the numerical scheme. From the geometrical standpoint, if $t_{\text{ref}} = t_{n+1/2}$ equation (7) (with $\theta = 1/2$) may be viewed as centered discretization along the characteristics. On the other hand, for $t_{\text{ref}} = t_{n+1}$ we move backwards. This is relative to the particle we follow—in both cases, though, the discretization is formally of second order (see Figure 1).

Let us consider the case $t_{\text{ref}} = t_{n+1}$ and so $\tilde{\mathbf{x}}(t_{n+1}) = \mathbf{x}_{\text{ref}}$ (see equations (4)). To emphasize that $\mathbf{x}_{\text{ref}}$ is arbitrary, we shall write $\mathbf{x}$ instead of $\mathbf{x}_{\text{ref}}$.

The solution of problem (4) may be approximated up to second order as follows:

$$\begin{aligned}
\tilde{\mathbf{x}}(t_n) &= \tilde{\mathbf{x}}(t_{n+1}) - \Delta t\, \mathbf{u}(\tilde{\mathbf{x}}(t_{n+1}), t_n) + O(\Delta t^2) \\
&= \mathbf{x} - \Delta t\, \mathbf{u}^n + O(\Delta t^2)
\end{aligned} \tag{8}$$

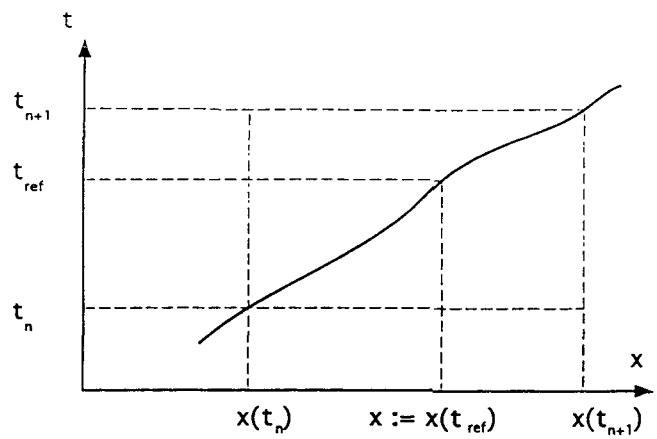

Figure 1. Reference particle for the discretization along the characteristics

and therefore,

$$\mathbf{u}(\tilde{\mathbf{x}}(t_n), t_n) = \mathbf{u}(\mathbf{x} - \Delta t\, \mathbf{u}^n + O(\Delta t^2), t_n)$$

$$= \mathbf{u}^n - \Delta t\, u_i^n \frac{\partial \mathbf{u}^n}{\partial x_i} + O(\Delta t^2) \tag{9}$$

Equation (9) allows to obtain the following third-order approximation to the trajectory $\tilde{\mathbf{x}}$:

$$\tilde{\mathbf{x}}(t_n) = \tilde{\mathbf{x}}(t_{n+1}) - \frac{\Delta t}{2}\left[\mathbf{u}(\tilde{\mathbf{x}}(t_{n+1}), t_{n+1}) + \mathbf{u}(\tilde{\mathbf{x}}(t_n), t_n)\right] + O(\Delta t^3)$$

$$= \mathbf{x} - \Delta t\, \mathbf{u}^{n+1/2} + \frac{\Delta t^2}{2} u_i^n \frac{\partial \mathbf{u}^n}{\partial x_i} + O(\Delta t^3) \tag{10}$$

where $\mathbf{u}^{n+1/2} = [\mathbf{u}^{n+1} + \mathbf{u}^n]/2$. Using the approximation (10) we obtain

$$\phi(\tilde{\mathbf{x}}(t_n), t_n) = \phi\left(\mathbf{x} - \Delta t\, \mathbf{u}^{n+1/2} + \frac{\Delta t^2}{2} u_i^n \frac{\partial \mathbf{u}^n}{\partial x_i} + O(\Delta t^3), t_n\right)$$

$$= \phi^n - \Delta t\, u_j^{n+1/2} \frac{\partial \phi^n}{\partial x_j} + \frac{\Delta t^2}{2} u_i^n \frac{\partial u_i^n}{\partial x_i} \frac{\partial \phi^n}{\partial x_j}$$

$$+ \frac{\Delta t^2}{2} u_i^{n+1/2} u_j^{n+1/2} \frac{\partial}{\partial x_i} \frac{\partial \phi^n}{\partial x_j} + O(\Delta t^3) \tag{11}$$

Using the fact that $\mathbf{u}^{n+1/2} = \mathbf{u}^n + O(\Delta t)$ we may write equation (11) as

$$\phi(\tilde{\mathbf{x}}(t_n), t_n) = \phi^n - \Delta t\, u_j^{n+1/2} \frac{\partial \phi^n}{\partial x_j} + \frac{\Delta t^2}{2} u_i^n \frac{\partial}{\partial x_i}\left(u_j^n \frac{\partial \phi^n}{\partial x_j}\right) + O(\Delta t^3) \tag{12}$$

This holds for any function ϕ. A simplified version of this approximation is

$$\phi(\tilde{\mathbf{x}}(t_n), t_n) = \phi^n - \Delta t\, u_j^n \frac{\partial \phi^n}{\partial x_j} + O(\Delta t^2) \tag{13}$$

Using equation (12) in the discretization of the temporal derivative in equation (7) (with $\theta = 1/2$) and equation (13) to approximate the rest of the terms evaluated at $\mathbf{x} = \tilde{\mathbf{x}}(t_n)$ and $t = t_n$ we finally obtain

$$\frac{1}{\Delta t}[\phi^{n+1} + \phi^n] + u_j^{n+1/2} \frac{\partial \phi^n}{\partial x_j} - \frac{\partial}{\partial x_i}\left(k \frac{\partial \phi^{n+1/2}}{\partial x_i}\right) + a\phi^{n+1/2} + Q^{n+1/2}$$

$$- \frac{\Delta t}{2} u_k^n \frac{\partial}{\partial x_k}\left[u_i \frac{\partial \phi^n}{\partial x_i} - \frac{\partial}{\partial x_i}\left(k \frac{\partial \phi^n}{\partial x_i}\right) + a\phi^n + Q^n\right] = 0 \tag{14a}$$

Once this semi-discrete problem has been obtained, we may further approximate values at the time level $n + 1/2$ by values at n, thus obtaining a fully explicit scheme. This involves only an approximation of the temporal argument of the functions. Moreover, if we use linear elements for the space discretization the diffusion term in the last bracketed term of equation (14a) may be neglected, assuming that it is evaluated elementwise. Using these approximations, we obtain

$$\frac{1}{\Delta t}[\phi^{n+1} - \phi^n] + u_j^n \frac{\partial \phi^n}{\partial x_j} - \frac{\partial}{\partial x_i}\left(k \frac{\partial \phi^n}{\partial x_i}\right) + a\phi^n + Q^n$$

$$- \frac{\Delta t}{2} u_k^n \frac{\partial}{\partial x_k}\left[u_i \frac{\partial \phi^n}{\partial x_i} + a\phi^n + Q^n\right] = 0 \tag{14b}$$

This can also be written as

$$\Delta\phi = \Delta t\left[-\frac{\partial}{\partial x_j}(u_j\phi) + \frac{\partial}{\partial x_j}\left(k\frac{\partial\phi}{\partial x_j}\right) - Q\right]^n + \frac{\Delta t^2}{2}\left[u_i\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(u_j\phi) + Q\right)\right]^n \tag{15}$$

where $\Delta\phi = \phi^{n+1} - \phi^n$. This is the form of the characteristic Galerkin method that will be used throughout the paper. Note however that the diffusion contribution must be added in the last term of equation (15) if quadratic elements are used.

An identical expression can be here derived by using a higher order time approximation of the Lax–Wendroff type for a single scalar variable. However, and as already mentioned, since equation (15) is derived from a self-adjoint problem in space, the spatial discretization by Galerkin method is optimal. We can write thus the approximation

$$\phi = \mathbf{N}\bar{\phi} \tag{16}$$

and use the weighting $\mathbf{N}^{\mathrm{T}}$ in the integrated residual expression. Thus we obtain

$$\mathbf{M}(\bar{\phi}^{n+1} - \bar{\phi}^n) = -\Delta t[(\mathbf{C}\bar{\phi}^n + \mathbf{K}\bar{\phi}^n + \mathbf{f}^n) + \Delta t(\mathbf{K}_u\bar{\phi}^n + \mathbf{f}_s^n)] \tag{17}$$

where

$$\mathbf{M} = \int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Omega, \quad \mathbf{C} = \int_\Omega \mathbf{N}^{\mathrm{T}}\frac{\partial(u_i\mathbf{N})}{\partial x_i}\,\mathrm{d}\Omega$$

$$\mathbf{K} = \int_\Omega \frac{\partial\mathbf{N}^{\mathrm{T}}}{\partial x_i}k\frac{\partial\mathbf{N}}{\partial x_i}\,\mathrm{d}\Omega + b.t. \quad \mathbf{f} = \int_\Omega \mathbf{N}^{\mathrm{T}}Q\,\mathrm{d}\Omega + b.t. \tag{18a}$$

Expressions for $\mathbf{K}_u$ and $\mathbf{f}_s^n$ come from the new term introduced by the discretization along the characteristics. After integration by parts, the expression of $\mathbf{K}_u$ and $\mathbf{f}_s$ is

$$\mathbf{K}_u = \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_i}(u_i\mathbf{N}^{\mathrm{T}})\frac{\partial}{\partial x_j}(u_j\mathbf{N})\,\mathrm{d}\Omega$$

$$\mathbf{f}_s = \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_i}(u_i\mathbf{N}^{\mathrm{T}})Q\,\mathrm{d}\Omega + b.t. \tag{18b}$$

The approximation is valid for any scalar-convected quantity even if that is one of the components of the velocity $\mathbf{u}$ itself, as is the case with the momentum conservation equations. For this reason we have elaborated the full details of the spatial approximation as the matrices will be repeatedly used.

It is of interest to remark that the explicit form of the equation (17) is conditionally stable. For one-dimensional problems using linear elements, the stability condition is given as (neglecting the effect of sources)

$$\Delta t \leqslant \Delta t_{\mathrm{crit}} = \frac{h}{|u|}\left(\sqrt{\frac{1}{P^2} + \frac{1}{3}} - \frac{1}{P}\right) \tag{19a}$$

In above the Peclet number P is defined as

$$P = \frac{|\mathbf{u}|h}{k} \tag{19b}$$

With h being the representative element size. The term 1/3 in equation (19a) may be replaced by 1 if mass lumping is used. Concerning quadratic elements, the stability limit for 1-D problems can be obtained as described in.[32]

In 2-D problems the critical time-step size may be computed as[32]

$$\Delta t_{\text{crit}} = \frac{\Delta t_\sigma \Delta t_v}{\Delta t_\sigma + \Delta t_v} \tag{19c}$$

where Δt_σ is given by (19a) and $\Delta t_v = h^2/2k$ is the diffusive limit for the critical one-dimensional time-step size.

Further, with $\Delta t = \Delta t_{\text{crit}}$ the steady-state solution results in an (almost) identical diffusion change to that obtained by using the optimal streamline upwinding procedures.[6] Thus, if steady-state solutions are the main objective of the computation such a value of Δt should be used in connection with the $\mathbf{K}_u$ term.

3. THE GENERAL FRACTIONAL STEP ALGORITHM FOR THE NAVIER–STOKES EQUATIONS

3.1. The equations of flow

The full conservation form of the Navier–Stokes equations for compressible flow is traditionally written as

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} + \mathbf{Q} = 0 \tag{20}$$

with

$$\mathbf{V}^{\mathrm{T}} = [\rho, \rho u_1, \rho u_2, \rho u_3, \rho E] \tag{21a}$$

being the independent variable vector. Further,

$$\mathbf{F}_i^{\mathrm{T}} = [\rho u_i, \rho u_1 u_i + \delta_{1i} p, \rho u_2 u_i + \delta_{2i} p, \rho u_3 u_i + \delta_{3i} p, u_i(\rho E + p)] \tag{21b}$$

defines the convective flux vector and

$$\mathbf{G}_i^{\mathrm{T}} = \left[0, -\tau_{1i}, -\tau_{2i}, -\tau_{3i}, -k\frac{\partial T}{\partial x_i} - \tau_{ij} u_j\right] \tag{21c}$$

defines the diffusion flows. Finally

$$\mathbf{Q}^{\mathrm{T}} = [0, g_1, g_2, g_3, 0] \tag{21d}$$

gives the source terms due to (minus) the gravity acceleration.

In above the stress components τ_{ij} are related to velocity gradients by

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) \tag{22}$$

where ρ is the density, u_i are the velocity components, p is the pressure, E is the internal energy and T is the absolute temperature.

The equations are completed by the universal gas law

$$p = \rho R T \tag{23}$$

where R is the gas constant.

The sound velocity is defined assuming constant entropy as

$$c^2 = \frac{\partial p}{\partial \rho} = \frac{\gamma p}{\rho} \tag{24}$$

Further we can write conveniently

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} \tag{25}$$

though this expression assumes again constant entropy and is therefore only an approximation. We shall use in what follows equation (25), but later we will discuss the possibility of correcting any errors involved by an amendment of the algorithm. In any case, in explicit forms of the algorithm this approximation is not used.

While in compressible gas flow all the equations are fully coupled, for incompressible flow in which $c = \infty$ the energy equations can be solved independently after the velocity field has been established. Nevertheless a single algorithm for the solution of both problems is possible as we shall now show.

3.2. The general algorithm

For convenience we shall rewrite equation (20) in a more direct form, omitting initially the energy equation. These equations can be solved completely in a time increment Δt as the only coupling which exists is through the speed of sound c for which we shall simply use the value at time t_n due to the explicit nature of the time-stepping algorithm.

We thus write the first of equations (20), i.e. the mass flow continuity, as

$$\frac{\partial \rho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} = -\frac{\partial U_i}{\partial x_i} \tag{26}$$

in which we use equation (25). Further, for each of the momentum conservation equations we write similarly

$$\frac{\partial U_i}{\partial t} = -\frac{\partial f_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} - g_i \tag{27}$$

In above we define

$$U_i = \rho u_i \tag{28a}$$

and

$$f_{ij} = u_j(\rho u_i) = u_j U_i \tag{28b}$$

Since

$$\frac{\partial f_{ij}}{\partial x_j} = \frac{\partial u_j}{\partial x_j} U_i + u_j \frac{\partial U_i}{\partial x_j} \tag{29}$$

we can discretize in time equation (27) using the characteristic Galerkin process if the variation of p is known in time. Except for the pressure term this equation is similar to the convection–diffusion problem of equation (3). If we consider the pressure gradient evaluated at $t_n + \theta_2 \Delta t$, where $0 \leqslant \theta_2 \leqslant 1$, the term $(1 - \theta_2)\partial p/\partial x_i$ will be evaluated at time t_n and thus treated exactly as the source Q in equation (3). On the other hand, the term $\theta_2 \partial p/\partial x_i$ will be evaluated at $t_n + \Delta t$, so that no modification is needed for it to account for the discretization along the characteristics. We shall have therefore,

$$\begin{aligned} U_i^{n+1} - U_i^n = \Delta t \left[-\frac{\partial f_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} - g_i + \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}\left(\frac{\partial f_{ij}}{\partial x_j} + g_i\right)\right]^n \\ - (1 - \theta_2)\Delta t \left[\frac{\partial p}{\partial x_i} - \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}\frac{\partial p}{\partial x_i}\right]^n - \theta_2 \Delta t \left[\frac{\partial p}{\partial x_i}\right]^{n+1} \end{aligned} \tag{30}$$

Before proceeding further it is convenient to introduce an auxiliary variable $\tilde{U}_i$ such that

$$\Delta\tilde{U}_i = \tilde{U}_i^{n+1} - U_i^n = \Delta t\left[-\frac{\partial f_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} - g_i + \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}\left(\frac{\partial f_{ij}}{\partial x_j} + g_i\right)\right]^n \tag{31}$$

and therefore

$$\begin{aligned}\Delta U_i &= U_i^{n+1} - U_i^n \\ &= \Delta\tilde{U}_i - \Delta t(1-\theta_2)\left[\frac{\partial p}{\partial x_i} - \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}\frac{\partial p}{\partial x_i}\right]^n - \theta_2 \Delta t\left[\frac{\partial p}{\partial x_i}\right]^{n+1} \\ &= \Delta\tilde{U}_i - \Delta t\left[\frac{\partial p^n}{\partial x_i} + \theta_2 \frac{\partial \Delta p}{\partial x_i}\right] + (1-\theta_2)\frac{\Delta t^2}{2} u_k \frac{\partial}{\partial x_k}\frac{\partial p^n}{\partial x_i}\end{aligned} \tag{32}$$

where $\Delta p = p^{n+1} - p^n$ and the last term as before represents the 'source' correction.

From equation (26) we have, on omitting third-order terms

$$\begin{aligned}\Delta\rho = \left(\frac{1}{c^2}\right)^n \Delta p &= -\Delta t \frac{\partial U_i^{n+\theta_1}}{\partial x_i} \\ &= -\Delta t\left[\frac{\partial U_i^n}{\partial x_i} + \theta_1 \frac{\partial \Delta\tilde{U}_i}{\partial x_i} - \Delta t\,\theta_1\left(\frac{\partial^2 p^n}{\partial x_i \partial x_i} + \theta_2 \frac{\partial^2 \Delta p}{\partial x_i \partial x_i}\right)\right]\end{aligned} \tag{33}$$

It is clear that equations (31)–(33) can be solved after spatial discretization in the following order:

(1) equation (31) giving $\Delta\tilde{U}_i$,
(2) equation (33) giving Δp, and hence p_{n+1}
(3) equation (32) giving ΔU_i thus establishing the values of U_i at t_{n+1}.

In all of the equations given below the standard Galerkin procedure is used with the usual spatial discretization

$$U_i = \mathbf{N}\bar{\mathbf{U}}_i, \quad \Delta U_i = \mathbf{N}\Delta\bar{\mathbf{U}}_i, \quad \Delta\tilde{U}_i = \mathbf{N}\Delta\tilde{\mathbf{U}}_i \tag{34}$$

and

$$p = \mathbf{N}_p\bar{\mathbf{p}}$$

This gives from equation (31) the increment of $\tilde{\mathbf{U}}_i$ as

Step 1

$$\boxed{\Delta\tilde{\bar{\mathbf{U}}} = -\mathbf{M}^{-1}\Delta t[(\mathbf{C}\bar{\mathbf{U}} + \mathbf{K}\bar{\mathbf{U}} - \mathbf{f})\Delta t(\mathbf{K}_u\bar{\mathbf{U}} + \mathbf{f}_s)]^n} \tag{35}$$

where all the discretization matrices *are the same* as those defined by equation (18), although properly expanded to account for the vectorial character of the nodal variables.

Discretization of equation (33) gives similarly

Step 2

$$\boxed{(\tilde{\mathbf{M}} + \Delta t^2\,\theta_1\,\theta_2\mathbf{H})\Delta\bar{\mathbf{p}} = \Delta t[\mathbf{Q}(\bar{\mathbf{U}} + \theta_1\,\Delta\tilde{\bar{\mathbf{U}}}) - \Delta t\,\theta_1\mathbf{H}\bar{\mathbf{p}} - \mathbf{f}_p]^n} \tag{36}$$

which can be solved for $\Delta\bar{\mathbf{p}}$.

The new matrices arising here are

$$\mathbf{H} = \int_\Omega \left(\frac{\partial \mathbf{N}_p^T}{\partial x_i}\right) \frac{\partial \mathbf{N}_p}{\partial x_i} d\Omega$$

$$\tilde{\mathbf{M}} = \int_\Omega \mathbf{N}_p^T \left(\frac{1}{c^2}\right)^n \mathbf{N}_p d\Omega \tag{37}$$

$$\mathbf{Q} = \int_\Omega \frac{\partial \mathbf{N}_p^T}{\partial x_i} \mathbf{N} d\Omega$$

The question of establishing the boundary conditions for the pressure is discussed in detail in Appendix I.

The final stage of the computation of the mass flow vector $\mathbf{U}_i^{n+1}$ is completed by discretization of equation (32) and we have now simply

Step 3

$$\boxed{\Delta \bar{\mathbf{U}} = \Delta \tilde{\mathbf{U}} - \mathbf{M}^{-1} \Delta t [\mathbf{Q}^T (\bar{\mathbf{p}} + \theta_2 \Delta \bar{\mathbf{p}}) + \Delta t \mathbf{P} \bar{\mathbf{p}}]^n} \tag{38}$$

where

$$\mathbf{P} = \frac{1}{2}(1 - \theta_2) \int_\Omega \frac{\partial}{\partial x_i} (u_i \mathbf{N}_p^T) \frac{\partial \mathbf{N}_p}{\partial x_i} d\Omega \tag{39}$$

At the completion of this stage the values of $\bar{\mathbf{U}}_i^{n+1}$ and $\bar{\mathbf{p}}^{n+1}$ as fully determined but the computation of the energy ρE^{n+1} is needed so that new values of c^{n+1}, the speed of sound, can be determined.

The last equation in (20), i.e. the energy conservation equation, can be written as

$$\frac{\partial(\rho E)}{\partial t} = -\frac{\partial}{\partial x_i}(u_i \rho E) + \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) - \frac{\partial}{\partial x_i}(u_i p) + \frac{\partial}{\partial x_i}(\tau_{ij} u_j) \tag{40}$$

Once again this equation is identical in form to that of the scalar problem of equation (3) if we observe that p, U_i, etc. have been determined. Now the last term can be evaluated at time $(n + \theta_3)$, with $0 \leqslant \theta_3 \leqslant 1$, for improved accuracy, but in what follows we shall take $\theta_3 = 0$ for simplicity.

Using the characteristic Galerkin approximation for equation (40) and discretizing as

$$\rho E = \mathbf{N}_E \bar{\mathbf{E}}_v \tag{41}$$

we have

Step 4

$$\boxed{\Delta \bar{\mathbf{E}}_v = -\Delta t [\mathbf{C} \bar{\mathbf{E}}_v + \mathbf{K} \mathbf{T} + \mathbf{f}_e - \Delta t (\mathbf{K}_u \bar{\mathbf{E}}_v + \mathbf{f}_{es})]^n} \tag{42}$$

where $\bar{\mathbf{E}}_v$ contains the nodal values of ρE and again the matrices are identical to those previously obtained (assuming that ρE and T can be suitably scaled in the conduction term).

Again the forcing vectors can be appropriately defined as

$$\mathbf{f}_e = \int_\Omega \mathbf{N}_E^T \frac{\partial}{\partial x_j}(u_j p + \tau_{ij} u_i) d\Omega + b.t.$$

$$\mathbf{f}_{es} = -\frac{1}{2} \int_\Omega \frac{\partial}{\partial x_i}(u_i \mathbf{N}_E^T) \frac{\partial}{\partial x_j}(u_j p + \tau_{ij} u_i) d\Omega + b.t. \tag{43}$$

It is of interest to observe that the process of Step 4 can be extended in an identical manner to equations describing the transport of such quantities as turbulence parameters, chemical concentrations etc. once the first essential Steps 1–3 have been completed.

4. SEMI-IMPLICIT AND EXPLICIT FORMS OF THE ALGORITHM

The algorithm described can be used in a semi-implicit form and indeed only in this form incompressible problems in which $c = \infty$ and $\tilde{\mathbf{M}} = 0$ can be solved. Taking

$$\tfrac{1}{2} \leqslant \theta_1 \leqslant 1, \quad \tfrac{1}{2} \leqslant \theta_2 \leqslant 1 \tag{44}$$

the algorithm is conditionally stable. The permissible time step is governed by the critical step of the explicit relation solved in Step I of the algorithm. This is the standard convection–diffusion problem discussed in Section 2 and the same stability limits apply, reaching for an inviscid fluid a value close to

$$\Delta t_{\text{crit}} = \frac{h}{|\mathbf{u}|} \tag{45}$$

For slighty compressible or incompressible problems in which $\tilde{\mathbf{M}}$ is small or zero the semi implicit form is efficient and it should be noted that the matrix $\mathbf{H}$ of equation (37) does not vary during the computation process and can be partially inverted, thus simplifying the computations considerably.

In other semi-implicit forms when compressibility exists the question of the correctness of the approximation of equation (25) remains still unanswered and has to be further investigated. Here of course an iterative correction can be used. It should however be remarked that for low, subsonic speeds the approximation is nearly exact as the coupling with temperature changes is low. This is fortunate as it is only in that range of flows in which the semi-implicit form is advantageous, because at high supersonic speeds the critical time step of both semi-implicit and explicit schemes is very similar, and the latter provides a cheaper algorithm.

The fully explicit form is obtained by putting $\theta_2 = 0$. Now of course the critical step will be reduced to the order of

$$\frac{h}{c + |\mathbf{u}|} \tag{46}$$

and this is indeed the same limit as that encountered in other explicit forms of Euler or Navier–Stokes computational schemes currently effectively used.

The four equations (35), (36), (38) and (42) can be solved simultaneously if we take $\theta_2 = 0$ and the term $\Delta t\, \theta_1\, \Delta\tilde{\bar{\mathbf{U}}}$ in the R.H.S. of equation (36) is omitted. This of course is an additional approximation and is not necessary but is here introduced to mimic artificial diffusions previously extensively used with the standard Galerkin form.

Further the use of the approximation of equation (25) is now no longer necessary as the density increment is directly obtained if we note that

$$\tilde{\mathbf{M}}\, \Delta\bar{\mathbf{p}} \equiv \mathbf{M}\, \Delta\bar{p} \tag{47}$$

With above simplifications we can return to the original equations (20) and using the Galerkin approximation on these we can write directly

$$\Delta\bar{\mathbf{V}} = -\mathbf{M}^{-1}\, \Delta t \left[\int_\Omega \mathbf{N}^{\mathrm{T}} \left(\frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} \right) \mathrm{d}\Omega - \frac{1}{2}\, \Delta t \int_\Omega \mathbf{N}^{\mathrm{T}} \mathbf{D}\, \mathrm{d}\Omega \right]^n \tag{48}$$

omitting the source and boundary terms for clarity. The added diffusion terms **D** arising from the proposed algorithm are defined below and have to be integrated by parts in the usual manner.

$$\mathbf{D} = \begin{Bmatrix} 2\theta_1 \dfrac{\partial^2}{\partial x_i \partial x_i} p \\ u_i \dfrac{\partial}{\partial x_i}\left[\dfrac{\partial}{\partial x_j}(u_j \rho u_1) + \dfrac{\partial p}{\partial x_1}\right] \\ u_i \dfrac{\partial}{\partial x_i}\left[\dfrac{\partial}{\partial x_j}(u_j \rho u_2) + \dfrac{\partial p}{\partial x_2}\right] \\ u_i \dfrac{\partial}{\partial x_i}\left[\dfrac{\partial}{\partial x_j}(u_j \rho u_3) + \dfrac{\partial p}{\partial x_3}\right] \\ u_i \dfrac{\partial}{\partial x_i}\left[\dfrac{\partial}{\partial x_j}(u_j \rho E + u_j p)\right] \end{Bmatrix} \tag{49}$$

The 'diffusions' added are simple and largely streamline oriented (for divergence-free velocities) thus not masking the true effects of viscosity as happens in some schemes. The importance of the various terms will be discussed in Part II of this paper where detailed comparisons with other explicit schemes are made.

If steady-state results only are sought it would appear that Δt in the definition of the matrix **D** should be set at its optimal value of $\Delta t_{\text{crit}} \approx h/|\mathbf{u}|$ for inviscid flows or given by (19a) generally.

However the oversimplified scheme of equation (49) looses some accuracy and even when steady state is reached will give slightly different results than those obtained using the full sequential updating. The additional cost involved in computing the sequence $\Delta\tilde{\mathbf{U}} \rightarrow \Delta\mathbf{p} \rightarrow \Delta\bar{\mathbf{U}} \rightarrow \Delta\bar{\mathbf{E}}_v$ will have to be balanced against the accuracy increase.

It is of interest to note here in passing that the full sequential scheme introduces a so called 'fourth-order' diffusion proportional to $\Delta t\, \mathbf{Q}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{Q}\bar{\mathbf{p}}$ in addition to the second-order diffusion proportional to $\Delta t\, \mathbf{H}\bar{\mathbf{p}}$ into the computation. We shall indicate how this arises in the Section 6.

5. SHOCK CAPTURING TECHNIQUE

Once the basic formulation has been established, let us consider now the problem of removing the local oscillations that still remain using the characteristic Galerkin method. The shock capturing technique that we describe in this section is based on the application of the method described in References 27 and 28 for the convection–diffusion equation to the compressible Navier–Stokes equations. Another shock capturing method that we use is described in Reference 33.

The basic idea of the method that we want to describe now is to introduce an anisotropic additional diffusion to the discrete equations. This diffusion is taken proportional to the spatial residual of the previous time step, thus keeping the consistency of the finite element formulation when the steady state is reached.

Let us describe the method for the momentum equations (30). For simplicity, we shall consider the case $\theta_2 = 0$, although what follows can be easily extended to the case $\theta_2 > 0$. Introducing the spatial residual

$$R_i = \frac{\partial f_{ij}}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} + g_i \tag{50}$$

Equations (30) may be written as

$$U_i^{n+1} - U_i^n = -\Delta t\left[R_i - \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}(R_i)\right]^n \tag{51}$$

We now introduce for each element the shock capturing viscosity[27-31]

$$\nu_{SC} = \beta h \frac{|R_i|}{|\nabla U_i|}, \tag{52}$$

where h is the element length and ∇U_i is the gradient of U_i. If this gradient is very small, $\nu_{SC} = 0$ is taken. Also, when the viscosity is small, the parameter β in equation (52) may be taken as constant. The values 0·3 and 0·15 are effective for linear and quadratic elements, respectively. See Reference 27 for further discussion about the selection of β. All the values in equation (52) are understood to be evaluated for each element and at the nth time step. For the particular case of linear interpolation, derivatives of order two or higher can be neglected.

In principle, the shock capturing viscosity ν_{SC} could be introduced to equation (51). However, the second term within the brackets in this equation already introduces a viscosity along the streamlines of magnitude

$$\nu_{cg} = \frac{\Delta t}{2}|\mathbf{u}|^2 \tag{53}$$

To account for it, the additional viscosity that has to be introduced along the streamlines is only

$$\nu_{sl} = \max\{0, \nu_{SC} - \nu_{cg}\} \tag{54}$$

The final shock capturing viscosity is therefore anisotropic, of magnitude ν_{sl} along the streamlines and ν_{sc} along the other directions. Thus is schematically represented in Figure 2.

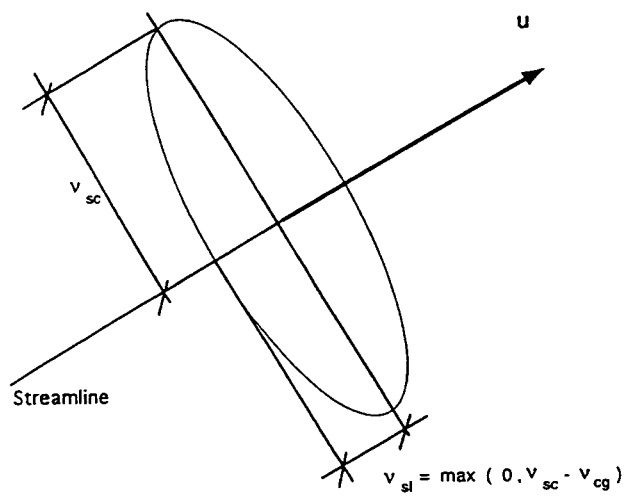

Figure 2. Shock capturing viscosity introduced along the streamlines and the normal directions

Once the anisotropic viscosity is introduced in equation (51) the final scheme will be

$$U_i^{n+1} - U_i^n = -\Delta t \left\{ R_i - \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}(R_i) - \frac{\partial}{\partial x_k}\left[\nu_{SC}\left(\frac{1}{|\mathbf{u}|^2} u_k u_l \right) \frac{\partial U_i}{\partial x_l} + \nu_{SI} \frac{1}{|\mathbf{u}|^2} u_k u_l \frac{\partial U_i}{\partial x_l} \right] \right\}^n \tag{55}$$

This equation replaces equation (30) (for $\theta_2 = 0$) when shock capturing has to be used.

Since all the terms corresponding to the additional viscosity are treated explicitly, we keep them in the first step of the splitting procedure, that is, they are added to the R.H.S. of equation (31). They affect therefore the calculation of $\tilde{\mathbf{U}}$ but equation (32) is unaltered.

A similar procedure to that yielding equation (55) should be applied also to the energy equation, which is also of convection–diffusion type. For the continuity equation, if a shock capturing diffusion is also added it will be isotropic, since no additional viscosity has been previously added.

6. WHY THE BB RESTRICTIONS ARE CIRCUMVENTED

We examine here the structure of equations reached in steady conditions. For simplicity we shall consider here only the Stokes form of governing equations in which the convective terms disappear. Further we shall take the fluid as incompressible and thus uncouple the energy equations. Now the three steps of equations (35), (36) and (38) are written as

$$\Delta\tilde{\bar{\mathbf{U}}} = -\Delta t \mathbf{M}^{-1}[\mathbf{K}\bar{\mathbf{U}}^n - \mathbf{f}] \tag{56a}$$

$$\Delta\bar{\mathbf{p}} = \frac{1}{\Delta t \theta_1 \theta_2} \mathbf{H}^{-1}[\mathbf{Q}(\bar{\mathbf{U}}^n + \theta_1 \Delta\tilde{\bar{\mathbf{U}}}) - \Delta t \theta_1 \mathbf{H}\bar{\mathbf{p}}^n - \mathbf{f}_p] \tag{56b}$$

$$\Delta\bar{\mathbf{U}} = \Delta\tilde{\bar{\mathbf{U}}} - \Delta t \mathbf{M}^{-1}\mathbf{Q}^{\mathrm{T}}(\bar{\mathbf{p}}^n + \theta_2 \Delta\bar{\mathbf{p}}) \tag{56c}$$

In steady state $\Delta\bar{\mathbf{p}} = \Delta\bar{\mathbf{U}} = 0$ and eliminating $\Delta\tilde{\bar{\mathbf{U}}}$ we can write (dropping now the superscript n)

$$\mathbf{K}\bar{\mathbf{U}} + \mathbf{Q}^{\mathrm{T}}\bar{\mathbf{p}} = \mathbf{f} \tag{57}$$

from equations (56a) and (56c) and

$$\mathbf{Q}\bar{\mathbf{U}} + \theta_1 \Delta t \mathbf{Q}\mathbf{M}^{-1}\mathbf{Q}^{\mathrm{T}}\mathbf{p} - \Delta t \theta_1 \mathbf{H}\mathbf{p} - \mathbf{f}_p = 0 \tag{58}$$

from equations (56b) and (56c).

We finally have a system which can be written in a form

$$\begin{bmatrix} \mathbf{K} & \mathbf{Q}^{\mathrm{T}} \\ -\mathbf{Q} & \Delta t \theta_1 [\mathbf{H} - \mathbf{Q}\mathbf{M}^{-1}\mathbf{Q}^{\mathrm{T}}] \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{U}} \\ \bar{\mathbf{p}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{Bmatrix} \tag{59}$$

where $\mathbf{f}_1$ and $\mathbf{f}_2$ arise from the forcing terms.

This system has a non-zero diagonal which is proportional to Δt and which, as already mentioned, is very similar to the forms suggested by other reasoning.[24–26]

Further it will be immediately observed that if the additional simplification introduced in equation (48) is made to avoid the sequential operations, the term $\Delta t \mathbf{Q}\mathbf{M}^{-1}\mathbf{Q}^{\mathrm{T}}$ disappears. This term is however very useful adding a 'smoothing' by spreading the effect of jumps, etc. to a wider pattern of elements.

It can be easily verified that if the pressure gradient term is retained in equation (31) (which would seem to give a better approximation) the diagonal term of equation (59) is identically zero and the BB conditions are still necessary.

7. SUMMARY

The algorithm here introduced follows similar lines of reasoning as were used in previous attempts to derive the 'universal' algorithm. Details however are different and in particular the introduction of the 'characteristic Galerkin' diffusion is more direct and different. In further parts of this paper numerical tests will be made and accuracy tested in various applications.

Application of the identical procedure to shallow water equation is obvious and again the performance is excellent as observed in Reference 34 presented in the same volume.

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APPENDIX I

We shall consider in this appendix the boundary conditions to be imposed for equation (33). Observe first that equation (32) may be written as

$$\Delta U_i = \Delta \tilde{U}_i - \Delta t \frac{\partial p^{n+\theta_2}}{\partial x_i} \tag{60}$$

where, up to terms of second order in Δt,

$$\begin{aligned} p^{n+\theta_2}(\mathbf{x}) &= (1-\theta_2)p^n(\tilde{\mathbf{x}}) + \theta_2 p^{n+1}(\mathbf{x}) \\ &= (1-\theta_2)\left[p^n(\mathbf{x}) - \frac{\Delta t}{2} u_i \frac{\partial}{\partial x_i} p^n(\mathbf{x})\right] + \theta_2 p^{n+1}(\mathbf{x}) \end{aligned} \tag{61}$$

The pressure boundary conditions we use are the continuity of the normal component of the momentum equation at the boundary. It may be readily checked that this is equivalent to the verification of the normal component of equation (60) at the boundary, that is

$$n_i \Delta U_i = n_i \Delta \tilde{U}_i - \Delta t\, n_i \frac{\partial p^{n+\theta_2}}{\partial x_i} \tag{62}$$

Multiplying equation (33) by the matrix of pressure test functions $\mathbf{N}_p$ and integrating by parts we obtain, using equation (62),

$$\begin{aligned} \int_\Omega \left(\frac{1}{c^2}\right)^n \mathbf{N}_p^{\mathrm{T}} \Delta p \, \mathrm{d}\Omega = -\Delta t \Bigg[&- \int_\Omega \frac{\partial \mathbf{N}_p^{\mathrm{T}}}{\partial x_i} U_i^n \, \mathrm{d}\Omega + \int_{\partial\Omega} \mathbf{N}_p^{\mathrm{T}} n_i \, \mathrm{d}\Gamma \\ &+ \theta_1 \int_{\partial\Omega} \mathbf{N}_p^{\mathrm{T}} n_i \Delta U_i \, \mathrm{d}\Gamma - \theta_1 \int_\Omega \frac{\partial \mathbf{N}_p^{\mathrm{T}}}{\partial x_i} \Delta \tilde{U}_i^n \, \mathrm{d}\Omega \\ &+ \Delta t \theta_1 \int_\Omega \frac{\partial \mathbf{N}_p^{\mathrm{T}}}{\partial x_i} \frac{\partial p^{n+\theta_2}}{\partial x_i} \, \mathrm{d}\Omega \Bigg] \end{aligned} \tag{63}$$

from where it follows that the term $\mathbf{f}_p$ in equation (36) is given by

$$\mathbf{f}_p = -\int_{\partial\Omega} \mathbf{N}_p^{\mathrm{T}} n_i U_i^n - \theta_1 \int_{\partial\Omega} \mathbf{N}_p^{\mathrm{T}} n_i \Delta U_i \, \mathrm{d}\Gamma \tag{64}$$

This expression involves ΔU_i, which is unknown at the moment of solving the pressure equation. However, the second term in the R.H.S. of equation (64) may be neglected. This approximation is exact if $\mathbf{U}$ is prescribed on the whole boundary $\partial\Omega$. On the other hand, and in order to avoid the need to compute the boundary integral, in equation (63) we may rewrite the first two terms of the two terms of the R.H.S. as

$$-\int_\Omega \frac{\partial \mathbf{N}_p^{\mathrm{T}}}{\partial x_i} U_i^n \, \mathrm{d}\Omega + \int_{\partial\Omega} \mathbf{N}_p^{\mathrm{T}} n_i U_i^n \, \mathrm{d}\Gamma = \int_\Omega \mathbf{N}_p^{\mathrm{T}} \frac{\partial U_i^n}{\partial x_i} \, \mathrm{d}\Omega \tag{65}$$

Instead of equation (27) we shall have

$$(\tilde{\mathbf{M}} + \Delta t^2 \theta_1 \theta_2 \mathbf{H}) \Delta \bar{\mathbf{p}} = \Delta t [\tilde{\mathbf{Q}} \bar{\mathbf{U}}^n + \theta_1 \mathbf{Q} \, \Delta \tilde{\mathbf{U}} - \Delta t \theta_1 \mathbf{H} \bar{\mathbf{p}}]^n \tag{66}$$

where

$$\tilde{\mathbf{Q}} = -\int_\Omega \mathbf{N}_p^{\mathrm{T}} \frac{\partial \mathbf{N}}{\partial x_i} \, \mathrm{d}\Omega$$

Observe that no term $\mathbf{f}_p$ appears in equation (66).

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